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Block Relaxation Techniques for Finite-Element Elliptic Equations: An Example

Daniel L. Boley*
Seymour V. Parter**

*Computer Science Department, Stanford University, Serra House, Serra Street, Stanford, CA 94305.
**University of Wisconsin—Madison, Madison, WI 53706.

BLOCK RELAXATION TECHNIQUES FOR FINITE-ELEMENT ELLIPTIC EQUATIONS: AN EXAMPLE

by

Daniel L. Boley and Seymour V. Parter

ABSTRACT

Consider the Ritz-Galerkin equations for the numerical solution of the two-point boundary value problem

$$u'' = f, \; o \leq x \leq 1,$$

$$u(o) = u(1) = 0.$$

We consider Ritz-Galerkin subspaces of hermite cubic splines with equally spaced knots. These equations are then solved via iterative methods. The rate of convergence of these methods is estimated.

1. INTRODUCTION

In [1] Boley, Buzbee and Parter developed an approach for obtaining asymptotic formulas for the "rates of convergence" for some block iterative methods applied to the solution of the "model problem." That is, we consider the boundary value problem

$$(1.1) \qquad \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y^2} = f(x,y), \qquad (x,y) \in \Omega$$

$$(1.2) \qquad u(x,y) = g(x,y) \;, \qquad (x,y) \in \partial\,\Omega$$

where Ω is unit square $0 \leq x,y \leq 1$. The algebraic problem arises when Δ is replaced by Δ_h , the well known five point difference approximation. The work in [1] was based on the ideas developed in [4].

In view of the popularity of finite-element methods for the numerical solution of (1.1)-(1.2), it seems desirable to investigate the applicability of these ideas for those linear algebraic problems which arise in the finite-element problems.

In this preliminary report we consider the simplest two-point boundary value problem

$$(1.3) \qquad u''(x) = f(x) \;, \qquad 0 \leq x \leq 1$$

$$(1.4) \qquad u(0) = u(1) = 0 \;.$$

We discuss a Ritz-Galerkin method based on Hermite cubic splines. We then analyze two particular block iterative methods for the solution of the ensuing linear algebraic system. It is quite clear that this analysis can be extended to a large class of block iterative methods for the general two-point boundary value problem

$$(1.5) \qquad (p(x)u')' - q(x)u = f(x) \;, \qquad 0 \leq x \leq 1$$

$$u(0) = u(1) = 0 \quad . \tag{1.6}$$

However, in order to give a complete, clear discussion without unnecessary complications, we limit outselves to the simplest case.

2. THE SIMPLEST TWO-POINT BOUNDARY VALUE PROBLEM: FORMULATION

Consider the boundary value problem

$$\left(\frac{d}{dx}\right)^2 u(x) = f(x) \quad , \qquad 0 < x < 1 \tag{2.1}$$

$$u(0) = u(1) = 0 \quad . \tag{2.2}$$

In this section we describe the Ritz-Galerkin method based on Hermite cubic splines. While this has been done many times [5], [6], [7], it will be of some advantage to pinpoint certain basic facts.

Let an integer $N > 0$ be chosen and let $h = 1/N+1$ and let $x_k = kh$, $k = 0, 1, \ldots N+1$. Let $S(h)$ be the space of Hermite cubic splines on the knot sequence $\{x_k\}$ which satisfy (2.2). That is

$$S(h) \equiv \left\{ U(x) \varepsilon\ C^1[0,1],\ U(0) = U(1) = 0 \ ,\quad U(x)\Big|_{I_k} \varepsilon\ \Pi_4(I_k) \right\} \tag{2.3}$$

where I_k is the interval (x_k, x_{k+1}) and $\Pi_4(I_k)$ is the space of polynomials of order 4 (degree ≤ 3) defined on I_k .

Let $h_j(x)$ and $h_j^I(x)$ be given as in [5, Chap. 3]. These functions are a basis for $S(h)$. In fact, if $\phi(x) \varepsilon\ S(h)$ then

$$\phi(x) = \sum_{k=1}^{N} \phi(x_k)\ h_k(x) + \sum_{k=0}^{N+1} \phi'(x_k)\ h_k^I(x) \quad . \tag{2.4}$$

The Ritz-Galerkin equations for an approximant $U(x) \varepsilon\ S(h)$ to $u(x)$, the solution of (2.1), (2.2), are given by;

$$-\ (\phi', U') = (\phi, f) \quad \forall \quad \phi \varepsilon\ S(h) \tag{2.5a}$$

where

$$(g,v) = \int_0^1 g(x)\,\overline{v}(x)\,dx \quad . \tag{2.5b}$$

Letting ϕ run over the 2N+2 basis vectors $h_k(x)$, $h_k^1(x)$ we obtain 2N+2 unknowns $U(x_k)$, $U'(x_k)$. In particular, if we order the unknowns as follows

$$U'(x_0),\ U(x_1),\ U'(x_1),\ \ldots\ U(x_k),\ U'(x_k),\ \ldots\ U(x_N),\ U'(x_N),\ U'(x_{N+1}) \ , \tag{2.6a}$$

then the equations (2.5a) take the form

$$A(h)\,\hat{U} = \hat{f} \tag{2.6b}$$

where A(h) is best described as a block tridiagonal matrix (see [5, Chap. 7], [6, Sect. 1.7])

$$A(h) = \frac{1}{h}\left[C_{k-1}^T,\ B_k,\ C_k\right] \qquad k = 1,\ 2,\ \ldots\ N+2 \ , \tag{2.7a}$$

where

$$B_1 = B_{N+2} = 2h^2/15 \ , \tag{2.7b}$$

$$B_k = \begin{bmatrix} 12/5 & 0 \\ 0 & 4h^2/15 \end{bmatrix} \ , \quad k = 2,\ \ldots\ N+1 \ , \tag{2.7c}$$

$$C_1 = \begin{bmatrix} -\frac{h}{10} & -\frac{h^2}{30} \end{bmatrix} \ , \tag{2.7d}$$

$$C_k = \begin{bmatrix} -\frac{6}{5} & \frac{h}{10} \\ -\frac{h}{10} & -\frac{h^2}{30} \end{bmatrix} \qquad k = 2,\ 3,\ \ldots\ N \ , \tag{2.7e}$$

(2.7f)
$$C_{N+1} = \begin{bmatrix} h/10 \\ \\ -\frac{h^2}{30} \end{bmatrix} .$$

The vector $\hat{U}$ consists of the interpolation values $U(x_n)$, $U'(x_n)$ ordered as in (2.6a). The vector $\hat{f}$ is given by

(2.8)
$$\begin{cases} \hat{f}_1 = (f, h_0^I) \\ \left.\begin{array}{l} \hat{f}_{2k} = (f, h_k) \\ \hat{f}_{2k+1} = (f, h_k^I) \end{array}\right\} \quad k = 1, 2, \ldots N \\ \hat{f}_{2N+2} = (f, h_{N+1}^I) \end{cases}$$

The following facts are particularly useful. If $U(x)$, $V(x) \varepsilon S(h)$ and $\hat{U}$, $\hat{V}$ are the corresponding vectors of interpolation values - ordered as in (2.6a) - then

(2.9a)
$$\langle \hat{V}, A(h) \hat{U} \rangle = (V', U')$$

and

(2.9b)
$$\langle \hat{U}, \hat{f} \rangle = (U, f)$$

where $\langle \hat{U}, \hat{f} \rangle$ denotes the familiar vector inner product, i.e.,

(2.9c)
$$\langle \hat{U}, \hat{f} \rangle = \sum_{k=1}^{2N+2} \hat{U}_k \bar{\hat{f}}_k .$$

There is another important matrix, Q(h). This matrix is characterized by the fact that

$$Q(h)^* = Q(h)^T = Q(h)$$

and

(2.9d) $$\langle Q(h)\,\hat{U},\ \hat{V}\rangle = (U,\ V)$$

Once more, it is convenient to describe Q(h) as a block tridiagonal matrix. We have

(2.10a) $$Q(h) = \frac{h}{420}\left[E^T_{k-1}\ ,\ D_k\ ,\ E_k\right]\ ,\qquad k = 1,\ 2,\ \ldots\ N+2$$

where

(2.10b) $$D_1 = D_{N+2} = 4h^2\ ,$$

(2.10c) $$E_1 = [13h \quad -3h^2]\ ,$$

(2.10d) $$D_k = \begin{bmatrix} 312 & 0 \\ 0 & 8h^2 \end{bmatrix}\ ,\qquad k = 2,\ 3,\ \ldots\ N+1\ ,$$

(2.10e) $$E_k = \begin{bmatrix} 54 & -13h \\ 13h & -3h^2 \end{bmatrix}\ ,\qquad k = 2,\ 3,\ \ldots\ N\ ,$$

(2.10f) $$E_{N+1} = \begin{bmatrix} -13h \\ -3h^2 \end{bmatrix}\ .$$

3. THE ITERATIVE METHODS

To be consistent with the representation of A(h), Q(h) we partition the vector $\hat{U}$ as $\tilde{U}$ with

$$(3.1)\qquad \begin{cases} \tilde{U}_1 = \hat{U}_1 \\ \tilde{U}_k = \begin{bmatrix} \hat{U}_{2k-2} \\ \\ \hat{U}_{2k-1} \end{bmatrix} , \quad k = 2, 3, \dots N, N+1 \\ \tilde{U}_{N+2} = \hat{U}_{2N+2} \quad . \end{cases}$$

Then the equations (2.6b) may be written as

$$(3.2)\qquad B_k \tilde{U}_k = - C^T_{k-1} \tilde{U}_{k-1} - C_k \tilde{U}_{k+1} + \tilde{F}_k , \quad k = 1, 2, \dots N+2 \quad .$$

We use this representation to develop the block Jacobi and block SOR iterative schemes to solve these equations.

Let a guess $\tilde{U}^0$ be given. Then the block Jacobi iterates $\tilde{U}^{\nu+1}$ are the solutions of the problems

$$(3.3)\qquad B_k \tilde{U}^{\nu+1}_k = - C^T_{k-1} \tilde{U}^{\nu}_{k-1} - C_k \tilde{U}^{\nu}_k + \tilde{F}_k , \quad k = 1, 2, \dots N+2 \quad .$$

A related iterative procedure may be obtained as follows. Suppose N is even, say N = 2J . Let

$$\beta_k = \begin{bmatrix} B_{2k-1} & C_{2k-1} \\ \\ C^T_{2k-1} & B_{2k} \end{bmatrix} , \quad k = 1, 2, \dots J+1 \quad ,$$

$$\gamma_k = \begin{bmatrix} \bigcirc & \bigcirc \\ & \\ C_{2k} & \bigcirc \end{bmatrix}, \quad k = 1,\ 2,\ \ldots\ J+1\ ,$$

$$V_k = \begin{bmatrix} \tilde{U}_{2k-1} \\ \\ \tilde{U}_{2k} \end{bmatrix}, \quad G_k = \begin{bmatrix} \tilde{F}_{2k-1} \\ \\ \tilde{F}_{2k} \end{bmatrix}, \quad k = 1,\ 2,\ \ldots\ J+1\ .$$

The equation (2.6b) may also be written as

$$(3.2') \qquad \beta_k\, V_k = -\gamma^T_{k-1}\, V_{k-1} - \gamma_k\, V_{k+1} + G_k\ , \quad k = 1,\ 2,\ \ldots,\ J+1\ .$$

For this block representation the block Jacobi iterates $V^{\nu+1}$ are solutions of the problems

$$(3.3') \qquad \beta_k\, V^{\nu+1}_k = -\gamma^T_{k-1}\, V^{\nu}_{k-1} - \gamma_{k+1}\, V^{\nu}_{k+1} + G_k,\quad k = 1,\ 2,\ \ldots,\ J+1\ .$$

Given a parameter ω, the block successive over-relaxation (SOR) iterative schemes take the form

$$(3.4) \qquad B_k\, \tilde{U}^{\nu+1}_k = -\omega\ \ C^T_{k-1}\, \tilde{U}^{\nu+1}_{k-1} - \omega\ C_k\, \tilde{U}^{\nu}_{k+1} + (1-\omega)B_k\, \tilde{U}^{\nu}_k + \tilde{F}_k\ .$$

$$(3.4') \qquad \beta_k\, V^{\nu+1}_k = -\omega\ \gamma^T_{k-1}\, V^{\nu+1}_{k-1} - \omega\ \gamma_k\, V^{\nu}_{k+1} + (1-\omega)\beta_k\, V^{\nu}_k + G_k\ .$$

Since (3.3), (3.3') are each a block tridiagonal iteration which is a special case of block property A (see [7]) we know that: if $\rho = \rho(J)$ is the dominant eigenvalue of the iterative procedure (3.3), (3.3'), then the optimal $\omega = \omega_b$ is given by

(3.5) $$\omega_b = 1 + \left[\frac{\rho}{1+\sqrt{1-\rho^2}}\right]^2 \quad .$$

Moreover, the dominant eigenvalue $\rho(S)$ of the block SOR method is given by

(3.6) $$\rho(S) = \omega_b - 1 = \left[\frac{\rho}{1+\sqrt{1-\rho^2}}\right]^2$$

Thus, we are concerned with the dominant eigenvalue $\rho = \rho(J)$ of the eigenvalue problems

(3.7) $$\lambda\, B_k \tilde{U}_k + C^T_{k-1} \tilde{U}_{k-1} + C_k \tilde{U}_{k+1} = 0 \quad , \qquad k = 1, 2, \ldots N+2 \quad .$$

(3.7') $$\lambda\, \beta_k V_k + \gamma^T_{k-1} V_{k-1} + \gamma_k V_{k+1} = 0 \quad , \qquad k = 1, 2, \ldots J+1 \quad .$$

We are now able to state our basic estimates.

Theorem A: For the equation (3.7) we have

(3.8) $$\rho(J) = 1 - \frac{5}{12}\pi^2 h^2 + 0(h^3) \quad .$$

For the equation (3.7') we have

(3.8') $$\rho(J) = 1 - \frac{5}{6}\pi^2 h^2 + 0(h^3) \quad .$$

4. ESTIMATING $\rho = \rho(J)$

As in [1] we write the eigenvalue problem (3.7) in the following form. Let

(4.1) $$M = \text{diag}\left\{B_1, B_2, \ldots B_N, B_{N+1}, B_{N+2}\right\}$$

(4.2) $$N = -\left[C^T_{k-1}, 0, C^T_k\right] \quad , \qquad k = 1, 2, \ldots N+2 \quad .$$

Then

(4.3a) $$A(h) = M - N$$

and we are concerned with finding

$$\rho = \rho(J) = \max\left\{ |\lambda| \; ; \; (\lambda M - N)\tilde{U} = 0, \; \tilde{U} \neq 0 \right\} . \tag{4.3b}$$

Lemma 4.1: The number ρ is itself an eigenvalue and may be characterized by

$$\rho = \operatorname*{Max}_{\tilde{U}\neq 0} \frac{\langle N\tilde{U},\tilde{U}\rangle}{\langle M\tilde{U},\tilde{U}\rangle} \tag{4.4}$$

Proof: The matrix M is symmetric and positive definite while the matrix N is symmetric. Moreover, because of block property A (see [7]), if λ is an eigenvalue then so is $-\lambda$. Thus (4.4) follows from the classical Rayleigh characterization of such eigenvalues (see [2]).

Lemma 4.2: We have the following estimates

$$1 - \frac{5}{12}\pi^2 h^2 + 0(h^3) \leq \rho < 1 \quad . \tag{4.5}$$

Proof: Let $\tilde{U}$ be the eigenvector associated with ρ . Then

$$\rho = \frac{\langle N\tilde{U},\tilde{U}\rangle}{\langle M\tilde{U},\tilde{U}\rangle} > 0 \quad .$$

Hence $\langle N\tilde{U},\tilde{U}\rangle > 0$. However, A(h) is also positive definite (see (2.9a)) and

$$\rho = \frac{\langle N\tilde{U},\tilde{U}\rangle}{\langle A(h)\,\tilde{U},\tilde{U}\rangle + \langle N\tilde{U},\tilde{U}\rangle} < 1 \quad .$$

To obtain the left hand inequality of (4.5) we employ the test function: $\sin\pi x$. Of course $\sin\pi x \notin S(h)$, hence we use the interpolant. That is, let $U_0(x) \in S(h)$ and satisfy

$$U_0(x_k) = \sin\pi x_k \quad , \tag{4.6a}$$

$$U_0'(x_k) = \pi \cos \pi x_k \quad . \tag{4.6b}$$

Then,

$$\rho \geq \frac{\langle N\tilde{U}_0, \tilde{U}_0\rangle}{\langle M\tilde{U}_0, \tilde{U}_0\rangle} \quad .$$

An easy calculation now completes the proof.

Having obtained these bounds, we proceed as in [1]. Let $\tilde{U}$ be the eigenvector associated with ρ . Then

$$\rho\, M\, \tilde{U} = N\, \tilde{U}$$

$$\rho\, A(h)\, \tilde{U} = (1-\rho)\, N\, \tilde{U}$$

$$A(h)\, \tilde{U} = \left[\frac{1-\rho}{\rho h^2}\right] (h^2 N)\, \tilde{U} \quad .$$

We write

(4.7) $$A(h)\, \tilde{U} = \mu(h)\, \tilde{N}\, \tilde{U}$$

where

(4.7a) $$\mu(h) = \frac{1-\rho}{\rho h^2}$$

satisfies

(4.7b) $$0 < \mu(h) \leq \frac{5}{12}\pi^2 + O(h)$$

and

(4.7c) $$\tilde{N} = h^2\, N \quad .$$

Lemma 4.3: For every $U(x) \in S(h)$, let $\tilde{U}$ be the associated vector. Then, if $h \leq 1$ we have

$$(4.8) \qquad |\langle \tilde{N}\, \tilde{U}, \tilde{U} \rangle| \leq 2 \left\{ h \sum_{k=1}^{N} \left[U(x_k) \right]^2 + h^3 \sum_{k=0}^{N+1} U'(x_k) \ ^2 \right\} .$$

Moreover, if U(x) satisfies

$$(4.9a) \qquad |U(x_k) - U(x_{k+1})| \leq R\, h^{1/2}$$

$$(4.9b) \qquad |U'(x_k)| \leq R\, h^{-1/2}$$

then

$$(4.10a) \qquad \langle \tilde{N}\, \tilde{U}, \tilde{U} \rangle = \frac{12}{5} h \sum_{k=1}^{N} |U(x_k)|^2 + \delta(U)$$

where

$$(4.10b) \qquad |\delta(U)| \leq \frac{5}{3} R^2 h \quad .$$

<u>Proof</u>: A direct computation shows that

$$\begin{aligned}
\langle \tilde{N}\, \tilde{U}, \tilde{U} \rangle = {} & \frac{h^2}{5} U'(x_0)\, U(x_1) + \frac{h^3}{15} U'(x_0)\, U'(x_1) \\
& - \frac{h^2}{5} U'(x_{N+1})\, U(x_N) + \frac{h^3}{15} U'(x_N)\, U'(x_{N+1}) \\
& + \frac{12}{5} h \sum_{k=0}^{N} U(x_k)\, U(x_{k+1}) \\
& + \frac{h^2}{10} \sum_{k=1}^{N} U'(x_k) \left[U(x_{k+1}) - U(x_{k-1}) \right] \\
& + \frac{h^3}{15} \sum_{k=1}^{N-1} U'(x_k)\, U'(x_{k+1}) \quad .
\end{aligned}$$

Thus we obtain (4.8) from Schwarz's inequality.

Turning to the proof of (4.10), we see that the first and third terms above are together bounded by $\frac{1}{5} R^2 h$; so is the sixth term. The sum of the second, fourth, and last terms is bounded by $\frac{1}{15} R^2 h$. Finally we look at the fifth term. We note that

$$\sum_{k=0}^{N} U(x_k)\, U(x_{k+1}) = \frac{1}{2} \sum_{k=0}^{N} \left\{ \left[U(x_k)\right]^2 + \left[U(x_{k+1})\right]^2 \right\}$$

$$- \frac{1}{2} \sum_{k=0}^{N} \left[U(x_k) - U(x_{k+1})\right]^2 \quad .$$

The last term in this expression is bounded by $\frac{1}{2} R^2$ and the lemma is proven.

<u>Lemma 4.4</u>: For every $U(x) \varepsilon S(h)$ let $\tilde{U}$ be the associated vector. Suppose $U(x)$ satisfies (4.9a), (4.9b). Then

$$\langle Q(h)\, \tilde{U}, \tilde{U} \rangle = h \sum_{k=1}^{N} |U(x_k)|^2 + \sigma(U) \tag{4.11a}$$

where

$$|\sigma(U)| \leq R^2 h \quad . \tag{4.11b}$$

<u>Proof</u>: A direct computation shows that

$$\langle Q(h)\, \tilde{U},\tilde{U}\rangle = \frac{4}{420}\, h^3 \left\{ |U'(0)|^2 + |U'(1)|^2 \right\}$$

$$+ \frac{26}{420}\, h^2 \left[U'(0)\, U(x_1) - U'(1)\, U(x_N) \right]$$

$$- \frac{3}{420}\, h^3 \left[U'(0)\, U'(x_1) + U'(1)\, U'(x_N) \right]$$

$$+ \frac{8}{420}\, h^3 \sum_{k=1}^{N} |U'(x_k)|^2 - \frac{6}{420}\, h^3 \sum_{k=1}^{N-1} U'(x_k)\, U'(x_{k+1})$$

$$+ \frac{312}{420}\, h \sum_{k=1}^{N} |U(x_k)|^2 + \frac{108}{420}\, h \sum_{k=1}^{N} U(x_k)\, U(x_{k+1})$$

$$+ \frac{26}{420}\, h^2 \sum_{k=1}^{N} U'(x_k) \left[U(x_{k+1}) - U(x_{k-1}) \right]$$

The lemma now follows from the same pattern of proof as that given in lemma 4.3.

Corollary 4.4: If $U(x)$ satisfies (4.9a), (4.9b) then

$$\langle \tilde{N}\, \tilde{U},\tilde{U}\rangle = \frac{12}{5} \langle Q(h)\, \tilde{U},\tilde{U}\rangle + \delta(U) - \frac{12}{5}\, \sigma(U)$$

$$= \frac{12}{5} \int_0^1 |U(x)|^2\, dx + \delta(U) - \frac{12}{5}\, \sigma(U) \quad .$$

5. PROOF OF THEOREM A

We consider only (3.7) and (3.8). The arguments for (3.7') and 3.8') are essentially the same. Let $\tilde{U}$ be the eigenvector of (4.7). We know that $\langle \tilde{N}\, \tilde{U},\tilde{U}\rangle > 0$. So, we may normalize $\tilde{U}$ so that

(5.1) $$\langle \tilde{N}\, \tilde{U},\tilde{U}\rangle = 1 \quad .$$

Then (4.7) gives

$$(5.2) \qquad \langle \tilde{U}, A(h)\, \tilde{U} \rangle = \mu(h)\, \langle \tilde{U}, \tilde{N}\, \tilde{U} \rangle$$

That is, if h is small enough,

$$(5.3) \qquad \int_0^1 |U'(x)|^2\, dx = \langle \tilde{U}, A(h)\tilde{U} \rangle \le \pi^2 \quad .$$

Then,

$$|U(x) - U(y)| = \left|\int_x^y U'(t)\, dt\right| \le |x-y|^{1/2}\, \pi \quad .$$

Thus, (4.9a) holds with $R=\pi$. Moreover, as is well-known (see [2, p. 142]) there is a constant R_2 so that (5.3) implies

$$(5.4) \qquad |U'(x)| \le R_2\, h^{-1/2} \quad .$$

Applying corollary 4.4 we have

$$(5.5) \quad \int_0^1 |U'(x)|^2\, dx = \mu(h) \left[\frac{12}{5} \int_0^1 |U(x)|^2\, dx + \delta(U) - \frac{12}{5}\, \sigma(U) \right] ,$$

and

$$\int_0^1 |U(x)|^2\, dx = \frac{5}{12} + O(h) \quad .$$

Thus, we may rewrite (5.5) as

$$\mu(h) = \frac{5}{12} \frac{\int_0^1 |U'(x)|^2\, dx}{\int_0^1 |U(x)|^2\, dx} \Big(1 + O(h)\Big)$$

(5.6) $$\geq \frac{5}{12}\pi^2 \left(1 + O(h)\right) \quad .$$

This result, together with (4.7b) proves

(5.7a) $$\mu(h) = \frac{5}{12}\pi^2 + O(h) \quad ,$$

i.e.,

(5.7b) $$\rho(J) = 1 - \frac{5}{12}\pi^2 h^2 + O(h^3) \quad .$$

6. COMPUTATIONAL RESULTS

The following tables summarize our computational experience with this problem.

For the iteration (3.3). $\frac{5}{12} = 0.41666*$

h	Matrix Size	$\rho(J)$	$1 - (5/12)\pi^2 h^2$	$(1 - \rho)/\pi^2 h^2$
1/4	8	0.7606	0.74298	0.38810
1/8	16	0.9368	0.93574	0.40982
1/16	32	0.984005	0.983936	0.41488
1/32	64	0.995988	0.995984	0.41626
1/64	128	0.9989963	0.9989960	0.41655
1/128	256	0.99974903	0.99974900	0.41662

For the iteration (3.3'). $\frac{5}{6} = 0.8333*$

h	Matrix Size	$\rho(J)$	$(1 - \rho)/\pi^2 h^2$
1/7	14	0.846647518	0.76135
1/15	30	0.964236885	0.81530
1/31	62	0.991487474	0.82886
1/63	126	0.997930534	0.83222
1/127	254	0.999490238	0.83306
1/255	510	0.999873525	0.83327

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076-100	6.00	201-225	9.25	326-350	12.00	451-475	14.50	576-600	16.50
101-125	6.50	226-250	9.50	351-375	12.50	476-500	15.00	601-up	

Note: Add $2.50 for each additional 100-page increment from 601 pages up.